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Hybrid Randomness of Initial Imperfections and Axial Loading in Reliability of Cylindrical Shells

This study investigates the combined effect of randomness in initial geometric imperfections and the applied loading on the reliability of axially compressed cylindrical shells. In order to gain insight we consider simplest possible case when both the initial imperfections and the applied loads are uniformly distributed. It is shown that hybrid randomness may increase or decrease the reliability of the shell if the latter is treated, experiencing the sole randomness in initial imperfections. [DOI: 10.1115/1.4000412]

1 Introduction

The deterministic imperfection sensitivity has been developed by Koiter [1] in his classic Ph.D. thesis. It showed that the presence of the initial imperfections—deviations from intended shape can drastically reduce the load-carrying capacity of the cylindrical shell. Since then numerous articles have been written on this subject. The deterministic studies utilized the given analytical expressions for initial imperfections. However, it is realized that results for specified initial imperfections have no quantitative validity for shells with even slightly different imperfections. Therefore, the studies which postulate the initial imperfection function ought to be considered as merely qualitative illustrations of the phenomenon. The quantitative evaluation of the imperfection sensitivity demands for exact knowledge of the initial imperfections, in the form of measured profiles. Once measurements are conducted, it is realized that no two shells produced by the same manufacturing process exhibit exactly the same initial imperfections. Thus, the necessity to introduce some sort of uncertainty analysis can be well appreciated.

The concept of randomness of the initial imperfections—deviations from intended shape—was introduced by Bolotin [2] in his pioneering paper, in order to describe more realistically than the deterministic analysis, the behavior of cylindrical shells under axial compression. The topic was further developed in the books by Bolotin [3], Roorda [4], and Elishakoff et al. [5]. A detailed review of the recent research is described in papers by Chryssanthopoulos [6] and Elishakoff [7–9].

The randomness in the applied load was first introduced by Roorda [4] and followed by the study of Elishakoff [7]. Later on, Cederbaum and Arbocz [10], and Li et al. [11] studied this topic. The former investigation [10] utilized the first-order second moment method, whereas the latter work applied the conditional simulation technique [11].

In this paper we study the combined effect of random initial imperfections and random applied load. In order to grasp the effect, the simplest possible probability density—the uniform one—is taken in order to describe the random variables involved. Various cases are considered with evaluation of the attendant reliability, i.e., the probability that the buckling load exceeds the applied load. It is demonstrated that the claim that randomness in the loads reduces structural reliability must be modified. It turns out that the load reliability can be either detrimental or beneficial.

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2 Basic Equations

Koiter in his thesis [1] and a subsequent paper [2] provided the following relationship describing the imperfection sensitivity of a cylindrical shell:

$$(1-\rho)^2 - q\rho|\xi| = 0 \tag{1}$$

where

$$\rho = \lambda_s/\lambda_c \tag{2}$$

is the ratio between the buckling load λ_s of the imperfect structure and λ_c which represents the buckling load of the perfect structure, ξ is the nondimensional initial imperfection amplitude, and q is the numerical coefficient. Equation (1) in actuality represents the modification of Koiter's original formula, since in a very long shell if it is anticipated that initial imperfections either positive or negative will produce the same physical effect. It is seen from Eq. (1) that if the structure is perfect, i.e., $|\xi|=0$, then $\rho=1$ and there is no reduction in the load-carrying capacity. However, for $|\xi|>0$, the root $\rho<1$ describes the reduction of the load-carrying capacity because of initial imperfections.

In this paper we are interested in the case when both the applied load and the initial imperfections are random. Yet, it appears to be instructive to first investigate the case when the initial imperfection alone is treated as a random variable.

Hence, the initial imperfection ξ is treated as a random variable with given probability density $f(\xi)$ or cumulative distribution function F_ξ. We are interested in evaluating the structural reliability or the probability that the cylindrical shell will perform its interested mission.

It is not possible to demand the maximalistic task the shell not buckling, since it is the nature of the shell to undergo buckling. However, one can demand that the buckling will not occur prior prespecified nondimensional load α. Alternatively, we can identify the reliability with the probability that the shell buckles at loads exceeding α

$$R = \text{Prob}(\rho_s > \alpha) \tag{3}$$

In order to elucidate the imperfection sensitivity in the probabilistic context, we will first deal with the case when initial imperfections are uniformly distributed. Namely, three cases will be considered: (a) initial imperfections take positive values only, (b) initial imperfections assume negative values solely, and (c) initial imperfections take both positive and negative values.

3 Positive-Valued Uniformly Distributed Imperfections

Assume that initial imperfection's probability density reads as follows:

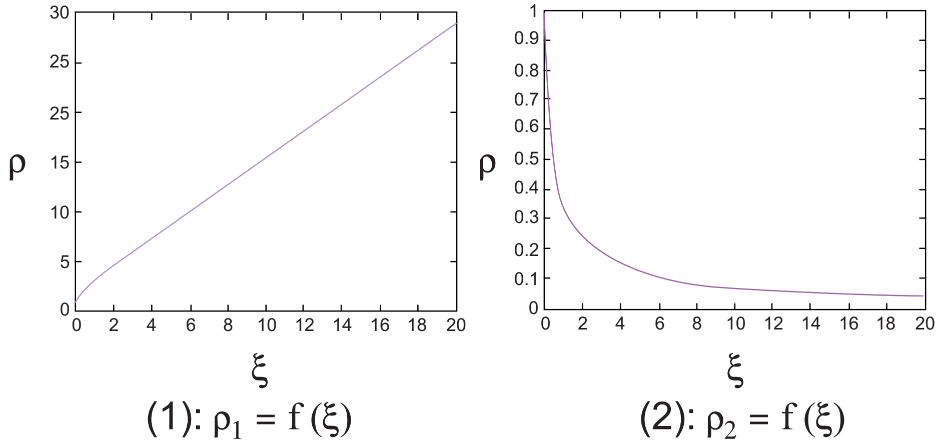

Fig. 1 Variation of ρ_1 and ρ_2 versus initial imperfection amplitude ξ

$$f(\xi) = \begin{cases} 0, & \text{for } \xi \le 0, \xi \ge \xi_0 \\ 1/\xi_0, & \text{for } 0 < \xi < \xi_0 \end{cases} \tag{4}$$

In order to evaluate the reliability $R(\alpha)$, we first express from Eq. (1) the value of ρ as a function of ξ:

$$\rho_{1,2} = (2 + q\xi \pm \sqrt{q\xi(4 + q\xi)})/2 \tag{5}$$

As is seen in Fig. 1, ρ_1 corresponds to values $\rho > 1$ and hence does not describe the physical phenomenon at hand. Only the curve ρ_2, with attendant values of $\rho < 1$, corresponds to the imperfection sensitivity. Thus, reliability becomes

$$R(\alpha) = \text{Prob}(\rho_s > \alpha) = \text{Prob}(\rho_2 > \alpha) = \text{Prob}[(2 + q\xi - \sqrt{q\xi(4 + q\xi)})/2 > \alpha] \tag{6}$$

leading to

$$R(\alpha) = \text{Prob}[\sqrt{q\xi(4 + q\xi)} < 2(1 - \alpha) + q\xi] \tag{7}$$

For $\alpha < 1$ and initial imperfection ξ taking positive values, the right-hand side $2\ (1-\alpha)+q\xi$ is positive. Therefore, the inequality in Eq. (7) can be replaced by

$$R(\alpha) = \text{Prob}[q\xi(4 + q\xi) < (2(1 - \alpha) + q\xi)^2] \tag{8}$$

After some algebra, we get

$$R = \text{Prob}[\xi < (1 - \alpha)^2/q\alpha] \tag{9}$$

or

$$R = F_\xi[(1 - \alpha)^2/q\alpha] \tag{10}$$

where F_ξ is the cumulative probability distribution function of initial imperfection, evaluated at $(1-\alpha)^2/q\alpha$. Therefore, the reliability becomes

$$R(\alpha) = \begin{cases} 0, & \text{for } (1 - \alpha)^2/q\alpha < 0 \\ (1 - \alpha)^2/q\alpha\xi_0, & \text{for } 0 < (1 - \alpha)^2/q\alpha \le \xi_0 \\ 1, & \text{for } (1 - \alpha)^2/q\alpha > \xi_0 \end{cases} \tag{11}$$

The first inequality is invalid. Consider the equality

$$(1 - \alpha)^2/q\alpha = \xi_0 \tag{12}$$

leading to

$$\alpha_{1,2} = (2 + q\xi_0 \pm \sqrt{q\xi_0(4 + q\xi_0)})/2 \tag{13}$$

The branch with the plus sign does not bear a physical sense since it is associated with consequence $\alpha > 1$. Hence only the branch with the minus sign has a physical sense. We denote

$$\alpha^* = (2 + q\xi_0 - \sqrt{q\xi_0(4 + q\xi_0)})/2 \tag{14}$$

Hence, the requirement $(1-\alpha)^2/q\alpha \le \xi_0$ in Eq. (11) is associated with an inequality $\alpha \ge \alpha^*$.

The reliability becomes

$$R(\alpha) = \begin{cases} 1, & \alpha \le \alpha^* \\ (1 - \alpha)^2/q\alpha\xi_0, & \alpha \ge \alpha^* \end{cases} \tag{15}$$

Let us deal with the design of the shell, demanding the structural reliability level to be at least r. The value of α corresponding to r is denoted hereinafter as $\alpha_{\text{allowable}}$. It satisfies the following equation:

$$r = \frac{(1 - \alpha_{\text{allowable}})^2}{q\alpha_{\text{allowable}}\xi_0} \tag{16}$$

For $\alpha_{\text{allowable}}$ we get

$$\alpha_{\text{allowable}\ 1,2} = (2 + qr\xi_0 \pm \sqrt{qr\xi_0(4 + qr\xi_0)})/2 \tag{17}$$

As is seen on Fig. 2, $\alpha_{\text{allowable},1}$ corresponds to values α in excess of unity and hence does not describe the physical phenomenon at hand. Thus, the final expression for $\alpha_{\text{allowable}}$ becomes

$$\alpha_{\text{allowable}} = \alpha_{\text{allowable},2} = (2 + qr\xi_0 - \sqrt{qr\xi_0(4 + qr\xi_0)})/2 \tag{18}$$

4 Combined Randomness in Imperfection and Load for Positive Imperfection Values

Let us extend these above considerations to the case when the applied load Λ is a random variable. Simplest possible case is the one when Λ is treated as a uniformly distributed random variable in the interval $[\lambda_1, \lambda_2]$, $\lambda_{j=1,2} \in [0,1]$, $\lambda_1 < \lambda_2$. Reliability in the new hybrid circumstances is denoted by R and becomes

$$R = \text{Prob}(\rho_s > \Lambda) \tag{19}$$

It is evaluated by the formula

$$R = \int_0^1 [1 - F_{\rho_s}(\alpha)] f_\Lambda(\alpha) d\alpha \tag{20}$$

or, since Λ varies in the interval $[\lambda_1, \lambda_2]$,

$$R = \int_{\lambda_1}^{\lambda_2} [1 - F_{\rho_s}(\alpha)] f_\Lambda(\alpha) d\alpha = \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} R(\alpha) d\alpha \tag{21}$$

The general shape of the reliability curve $R(\alpha)$ is depicted on Fig. 3. Five cases arise due to positions of λ_1 and λ_2 regarding α^* value. The reliability $R(\alpha)$ equals unity for $\alpha \le \alpha^*$, and decreases to zero, the latter value being achieved for $\alpha = 1$.

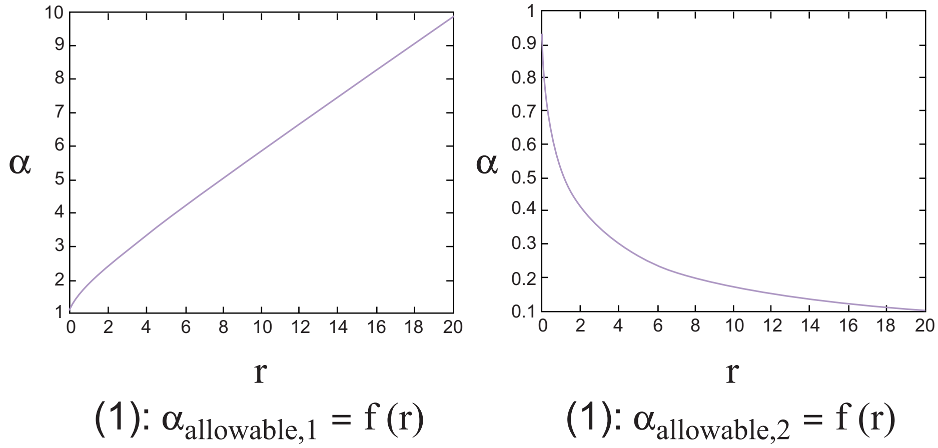

Fig. 2 Variation of $\alpha_{\text{allowable},1}$ and $\alpha_{\text{allowable},2}$ as function of r

For $\lambda_j \in [0, \alpha^*]$, where α^* is defined here as formally coinciding with Eq. (14), we get

$$R = \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} d\alpha = 1 \tag{22}$$

For $\lambda_1 \in [0, \alpha^*]$ and $\lambda_2 \in |\alpha^*, 1|$, we get

$$R = \frac{1}{\lambda_2 - \lambda_1} \left(\int_{\lambda_1}^{\alpha^*} d\alpha + \int_{\alpha^*}^{\lambda_2} \frac{(1-\alpha)^2}{q\alpha\xi_0} d\alpha \right) \tag{23}$$

or

$$R = \frac{1}{2q\xi_0(\lambda_2 - \lambda_1)} \left[2 \ln\left(\frac{\lambda_2}{\alpha^*}\right) + \lambda_2(\lambda_2 - 4) - \alpha^*(\alpha^* - 4) \right] + \frac{\alpha^* - \lambda_1}{\lambda_2 - \lambda_1} \tag{24}$$

For $\lambda_1 \in]\alpha^*, 1], \lambda_2 \in]\alpha^*, 1]$, we obtain

$$R = \frac{1}{\lambda_2 - \lambda_1} \int_{\lambda_1}^{\lambda_2} \frac{(1-\alpha)^2}{q\alpha\xi_0} d\alpha \tag{25}$$

or

$$R = \frac{1}{2q\xi_0(\lambda_2 - \lambda_1)} \left[2 \ln\left(\frac{\lambda_2}{\lambda_1}\right) + \lambda_2(\lambda_2 - 4) - \lambda_1(\lambda_1 - 4) \right] \tag{26}$$

To conduct numerical calculations, we set $\xi_0 = 0.3$ and $q = 4/3$; we obtain $\alpha^* = 0.5366$.

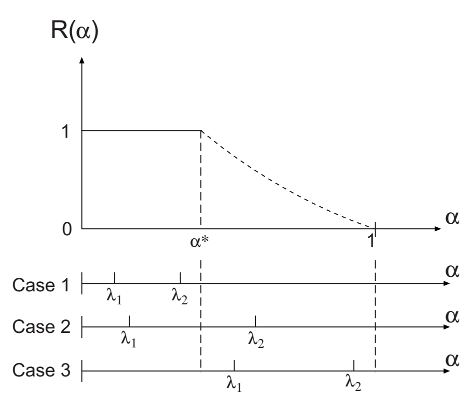

Fig. 3 Five possible different locations of load bounds with respect to α^* and unity

Then we calculate reliability values for the three cases introduced previously. The results are summarized in Table 1.The theoretical values of the reliability are denoted by R_{TH}. The calculation was also conducted by the crude Monte Carlo method with 10^7 simulations with the software MATLAB [12]. The results for reliability are denoted by R_{MC}. As is seen, an excellent agreement is present.

Calculations are also processed with PHIMECA [13] software for reliability analysis. Table 2 summarizes the results.

In Table 2, the index "form" implies the first-order approximation, "sormB" the second order parabolic approximation, "sormHB" the second order Hohenbichler and Breitung approximation, and "sormT" the second order Tvedt approximation. The latter one is seen to be the best with attendant percentage difference with exact solution being 0.08% for the second case and 3.23% for the third case.

Consider now the case when $\lambda_1 = 0$ and λ_2 numerically coincides with the value $\alpha_{\text{allowable}}$ found in Sec. 3, in Eq. (18). Reliability R_{TH}, given by the formula (20), is evaluated by the formula:

$$R = \int_0^{\alpha_{\text{allowable}}} [1 - F_{\rho_s}(\alpha)] f_\Lambda(\alpha) d\alpha \tag{27}$$

Table 1 Reliability values as depending upon bounds of the applied load

Case No.	1	2	3
λ_1	0.2	0.2	0.6
λ_2	0.3	0.6	0.9
R_{TH}	1	0.9722	0.2539
R_{MC}	1	0.9722	0.2539
Cov R_{MC} (%)	0	0.19	0.02

Table 2 Results obtained with PHIMECA SOFT in the case $\xi \geq 0$

Case No.	1	2	3
R_{form}	na	0.9532	0.2617
R_{sormB}	na	0.9699	0.2611
R_{sormHB}	na	0.9719	0. 2621
R_{sormT}	na	0.9730	0.2621
R_{TH}	1	0.9722	0.2539

na indicates "not applicable."

For $\xi_0=0.3$, $q=4/3$, and $r=0.9$, the calculations yield $\alpha^*=0.5366$ and $\alpha_{\text{allowable}}=0.5383$. As is seen, $\alpha_{\text{allowable}}$ is greater than α^*; hence, reliability expression corresponds to case 2, where $\lambda_1=0$ and $\lambda_2>\alpha^*$. Then Eq. (24) reduces to

$$R=\frac{1}{2q\xi_0\alpha_{\text{allowable}}}\left[2\ln\left(\frac{\alpha_{\text{allowable}}}{\alpha^*}\right)+\alpha_{\text{allowable}}(\alpha_{\text{allowable}}-4)-\alpha^*(\alpha^*-4)\right]+\frac{\alpha^*}{\alpha_{\text{allowable}}} \tag{28}$$

Finally, we get the value of reliability $R=0.999\,9848$, which signifies that as a result of the load randomness the actual reliability has *increased* above the level $r=0.99$ demanded for the case when the load was treated as a deterministic quantity. This conclusion on the increase of the structural reliability is not a universal one, however. If the random load takes values beyond the level $\alpha_{\text{allowable}}$, one can anticipate the decrease in structural reliability. It is instructive, therefore, to consider a case when the possible values of Λ are in the following interval $\lambda_1\le\lambda\le\lambda_2$. For example, $\lambda_1=\alpha_{\text{allowable}}$, $\lambda_2>\alpha_{\text{allowable}}$, and $\lambda_2=1.01\alpha_{\text{allowable}}$ can be chosen. In new circumstances, the reliability becomes

$$R=\int_{\lambda_1}^{\lambda_2}[1-F_{\rho_s}(\alpha)]f_\Lambda(\alpha)d\alpha=\frac{1}{\xi_0(\lambda_2-\lambda_1)}\int_{\lambda_1}^{\lambda_2}\frac{(\alpha-1)^2}{\alpha q}d\alpha \tag{29}$$

Final expression for the reliability reads

$$R=\frac{1}{2\xi_0(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\lambda_1}\right)+\lambda_2(\lambda_2-4)-\lambda_1(\lambda_1-4)\right] \tag{30}$$

In new circumstances, the actual reliability equals $R=0.98357$, which is *less* than $r=0.99$. We thus conclude that the load randomness can either increase or decrease the shell's reliability for the uniformly distributed initial imperfections and loads.

5 Negative-Valued Uniformly Distributed Imperfections

When initial imperfections take on only negative values, Eq. (1) reduces to

$$(1-\rho)^2+q\rho\xi=0 \tag{31}$$

leading to

$$\rho_{1,2}=(2-q\xi\pm\sqrt{q\xi(q\xi-4)})/2 \tag{32}$$

Only the branch with minus sign has a physical significance. The reliability becomes

$$R(\alpha)=\text{Prob}(\rho_2>\alpha)=\text{Prob}[(2-q\xi-\sqrt{q\xi(q\xi-4)})/2>\alpha] \tag{33}$$

or

$$R(\alpha)=\text{Prob}[\sqrt{q\xi(q\xi-4)}<2(1-\alpha)-q\xi] \tag{34}$$

For $\alpha<1$ and initial imperfection ξ taking negative values, the right-hand side $2(1-\alpha)-q\xi$ is positive. Therefore, the inequality in Eq. (8) can be replaced by

$$R(\alpha)=\text{Prob}[q\xi(q\xi-4)<(2(1-\alpha)-q\xi)^2] \tag{35}$$

After some derivations, we get

$$R(\alpha)=\text{Prob}[\xi<-(1-\alpha)^2/q\alpha] \tag{36}$$

or

$$R(\alpha)=F_\xi[-(1-\alpha)^2/q\alpha] \tag{37}$$

where F_ξ is the cumulative probability distribution function of initial imperfection, evaluated at $-(1-\alpha)^2/q\alpha$. $F_\xi(\xi)$ reads, in the region $\xi<0$:

Table 3 PHIMECA SOFT **results in the case $\xi\le 0$**

Case No.	1	2	3
R_{form}	na	0.9532	0.2617
R_{sormB}	na	0.9699	0.2613
R_{sormHB}	na	0.9719	0.2620
R_{sormT}	na	0.9730	0.2620

nc indicates "not applicable."

$$F_\xi(\xi)=\begin{cases}0, & \text{for } \xi>0\\ \xi/\xi_0, & \text{for } \xi_0<\xi\le 0\\ 1, & \text{for } \xi\le\xi_0\end{cases} \tag{38}$$

Therefore, the reliability becomes

$$R(\alpha)=\begin{cases}0, & \text{for } -(1-\alpha)^2/q\alpha>0\\ -(1-\alpha)^2/q\alpha\xi_0, & \text{for } \xi_0<-(1-\alpha)^2/q\alpha\le 0\\ 1, & \text{for } -(1-\alpha)^2/q\alpha\le\xi_0\end{cases} \tag{39}$$

The first inequality $-(1-\alpha)^2/q\alpha>0$ is invalid. Consider the equality

$$-(1-\alpha)^2/q\alpha=\xi_0 \tag{40}$$

resulting in

$$\alpha_{1,2}=(2-q\xi_0\pm\sqrt{q\xi_0(q\xi_0-4)})/2 \tag{41}$$

The branch with the plus sign does not bear a physical sense since it is associated with consequence $\alpha>1$. Hence only the branch with the minus sign has a physical sense. We denote

$$\alpha^*=(2-q\xi_0-\sqrt{q\xi_0(q\xi_0-4)})/2 \tag{42}$$

Hence the requirement reliability becomes

$$R(\alpha)=\begin{cases}1, & \alpha\le\alpha^*\\ -(1-\alpha)^2/q\alpha\xi_0, & \alpha\ge\alpha^*\end{cases} \tag{43}$$

However, since ξ_0 is negative, then by denoting $\xi_0=-X$, where X is a positive value, we arrive at

$$R(\alpha)=\begin{cases}1, & \alpha\le\alpha^*\\ (1-\alpha)^2/q\alpha X, & \alpha\ge\alpha^*\end{cases} \tag{44}$$

It is remarkable that the above expression for the reliability is exactly the same as Eq. (15). This means that whether ξ is positive or negative, the reliability of the structure is the same.

Thus, the analytical results obtained for negative ξ case are exactly the same, as the ones for positive ξ; $\alpha_{\text{allowable}}$ has the same expression as Eq. (18); reliability with the load being a random variable are given by the expressions written in Eqs. (22)–(25).

Numerical calculations are processed with PHIMECA SOFT [13] in the case of $\xi\le 0$ (Table 3). Input data are identical as calculations made for Table 2. The results are identical with three significant digits.

6 Uniformly Distributed Imperfections Taking on Both Positive and Negative Values

Let the initial imperfections take on both positive and negative values, $a\le\xi\le b$, where $a\le 0$ and $0\le b$. In the general case we should use Eq. (1) as a transfer function between imperfections and buckling loads:

$$(1-\rho)^2-q\rho|\xi|=0$$

We deduce the following load expression when ξ can take on either positive or negative values:

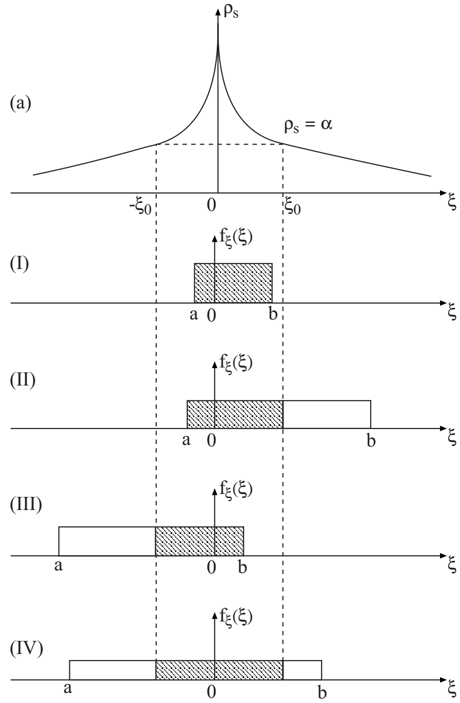

Fig. 4 Evaluation of reliability in four different cases

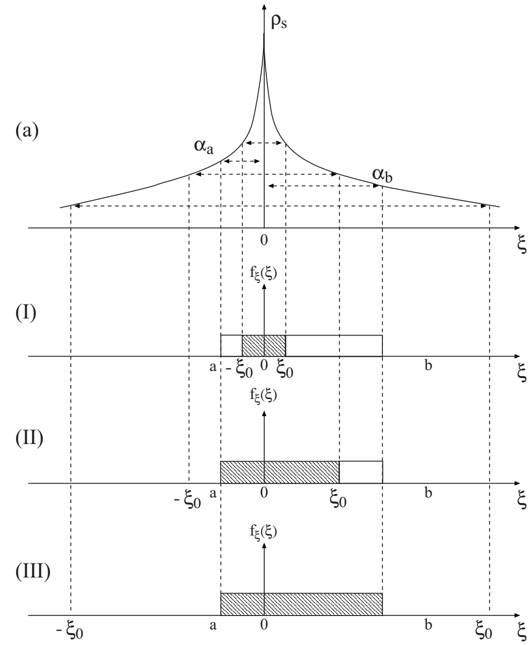

Fig. 5 Evaluation of reliability for different values of α

$$\rho=(2+q|\xi|-\sqrt{q|\xi|(4+q|\xi|)})/2 \tag{45}$$

Equation (46) coincides with expressions for ρ_2 in both Eqs. (6) and (33). Reliability becomes

$$R=\mathrm{Prob}(\rho>\alpha)=\mathrm{Prob}[(2+q|\xi|-\sqrt{q|\xi|(4+q|\xi|)})/2>\alpha] \tag{46}$$

or

$$R=\mathrm{Prob}[|\xi|<(1-\alpha)^2/q\alpha]=F_\xi[\xi_0]-F_\xi[-\xi_0] \tag{47}$$

where

$$\xi_0=(1-\alpha)^2/q\alpha \tag{48}$$

Figure 4 represents the variation of ρ_s with ξ. We fix the level $\rho_s=\alpha$, in order to calculate the probability that $\rho_s>\alpha$. The reliability equals then the probability that the initial imperfections take values between $-\xi_0$ and ξ_0, the values at which the horizontal line $\rho_s=\alpha$ crosses the function $\rho_s=\rho_s(\xi)$. Four different cases may occur.

The initial imperfection interval is fully enclosed in the interval $X=[-\xi_0,\xi_0]$.
The lower bound of initial imperfections belongs to the interval X, but the upper bound is outside it, i.e., $b>\xi_0$.
The lower bound of initial imperfections is outside X, i.e., $a<-\xi_0$, whereas the upper bound belongs to X.
Both a and b are outside X. These cases are depicted in Fig. 4. The hatched areas represent the structural reliability.

Reliability can be summarized as follows:

$$R(\alpha)=\begin{cases}1, & \text{for } \xi_0\geq \max(|a|,b)\\ \dfrac{\xi_0-a}{b-a}, & \text{for } |a|\leq\xi_0<b\\ \dfrac{b+\xi_0}{b-a}, & \text{for } b\leq\xi_0<|a|\\ \dfrac{2\xi_0}{b-a}, & \text{for } \xi_0<\min(|a|,b)\end{cases} \tag{49}$$

Reliability can be rewritten as function of α_a and α_b, which are loads corresponding to, respectively, bounds a and b. These expressions read

$$\alpha_a=(2-qa-\sqrt{qa(qa-4)})/2 \tag{50}$$

and

$$\alpha_b=(2+qb-\sqrt{qa(qb+4)})/2 \tag{51}$$

First, we consider $|a|<b$. Let us explore different values α_i that α can take for various $i=1,2,3,4$. The reliability equals the probability that the initial imperfections take values at which the horizontal lines $\rho_s=\alpha_i$ cross the function $\rho_s=\rho_s(\xi)$. Three different cases may occur, as depicted in Fig. 5. The first case represents $\alpha_a<\alpha$; the second one shows $\alpha_b<\alpha\leq\alpha_a$; and finally, the case $\alpha\leq\alpha_b$ is evaluated.

Reliability can be summarized as follows:

$$R(\alpha)=\begin{cases}1, & \text{for } \alpha\leq\min(\alpha_a,\alpha_b)\\ \dfrac{\xi_0-a}{b-a}, & \text{for } \alpha_b<\alpha\leq\alpha_a\\ \dfrac{2\xi_0}{b-a}, & \text{for } \max(\alpha_a,\alpha_b)<\alpha\end{cases} \tag{52}$$

Consider now the case $|a|>b$. In this case, $\alpha_b<\alpha_a$ and the condition $\alpha_b<\alpha\leq\alpha_a$ must be replaced by $\alpha_a<\alpha\leq\alpha_b$.We obtain

$$R(\alpha)=\begin{cases}1, & \text{for } \alpha\le \min(\alpha_a,\alpha_b)\\ \dfrac{b+\xi_0}{b-a}, & \text{for } \alpha_a<\alpha\le\alpha_b\\ \dfrac{2\xi_0}{b-a}, & \text{for } \max(\alpha_a,\alpha_b)<\alpha\end{cases}\tag{53}$$

Combining Eqs. (52) and (53) results in the following expression:

$$R(\alpha)=\begin{cases}1, & \text{for } \alpha\le \min(\alpha_a,\alpha_b)\\ \dfrac{b+\xi_0}{b-a}, & \text{for } \alpha_a<\alpha\le\alpha_b \text{ if } |a|>b\\ \dfrac{\xi_0-a}{b-a}, & \text{for } \alpha_b<\alpha\le\alpha_a \text{ if } |a|<b\\ \dfrac{2\xi_0}{b-a}, & \text{for } \max(\alpha_a,\alpha_b)<\alpha\end{cases}\tag{54}$$

Note that Eqs. (49) and (54) coincide, despite the fact that the conditions are written in a different form.

7 Combined Randomness in Imperfection and Load, for Either Positive or Negative Imperfection Values

Let us consider combined randomness in imperfection and load. Applied load Λ is defined as random variable in the interval $[\lambda_1,\lambda_2]$, $0<\lambda_1<\lambda_2$. Equation (20) takes the following form:

$$R=\int_{\lambda_1}^{\lambda_2}R(\alpha)f_\Lambda(\alpha)d\alpha=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}R(\alpha)d\alpha\tag{55}$$

To evaluate this reliability, consider $|a|>b$. We distinguish seven different cases

For $\lambda_j\in[0,\alpha_a]$, $(j=1,2)$ where α_a is defined here as formally coinciding with Eq. (50), we get

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}d\alpha=1\tag{56}$$

For $\lambda_1\in[0,\alpha_a]$ and $\lambda_2\in]\alpha_a,\alpha_b]$, we have

$$R\left|\begin{aligned}&=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_a}d\alpha+\int_{\alpha_a}^{\lambda_2}\frac{\xi_0+b}{b-a}d\alpha\right)\\&=\frac{\alpha_a-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{\lambda_2-\lambda_1}\int_{\alpha_a}^{\lambda_2}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}+\frac{b}{b-a}\right)d\alpha\end{aligned}\right.\tag{57}$$

or

$$R=\frac{\alpha_a-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\alpha_a}\right)+\lambda_2(\lambda_2-4)-\alpha_a(\alpha_a-4)\right]+\frac{b(\lambda_2-\alpha_a)}{(b-a)(\lambda_2-\lambda_1)}\tag{58}$$

For $\lambda_1\in[0,\alpha_a]$ and $\lambda_2>\alpha_b$,

$$R\left|\begin{aligned}&=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_a}d\alpha+\int_{\alpha_a}^{\alpha_b}\frac{\xi_0+b}{b-a}d\alpha+\int_{\alpha_b}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha\right)\\&=\frac{\alpha_a-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{\lambda_2-\lambda_1}\left[\int_{\alpha_a}^{\alpha_b}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}+\frac{b}{b-a}\right)d\alpha+\int_{\alpha_b}^{\lambda_2}\frac{2(1-\alpha)^2}{q\alpha(b-a)}d\alpha\right]\end{aligned}\right.\tag{59}$$

or

$$R=\frac{\alpha_a-\lambda_1}{\lambda_2-\lambda_1}+\frac{b(\alpha_b-\alpha_a)}{(b-a)(\lambda_2-\lambda_1)}+\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2^2}{\alpha_a\alpha_b}\right)+2\lambda_2(\lambda_2-4)-\alpha_a(\alpha_a-4)-\alpha_b(\alpha_b-4)\right]\tag{60}$$

For $\lambda_j\in]\alpha_a,\alpha_b]$, we obtain

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{\xi_0+b}{b-a}d\alpha=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}+\frac{b}{b-a}\right)d\alpha\tag{61}$$

or

$$R=\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\lambda_1}\right)+\lambda_2(\lambda_2-4)-\lambda_1(\lambda_1-4)\right]+\frac{b}{(b-a)}\tag{62}$$

For $\lambda_1\in[\alpha_a,\alpha_b]$ and $\lambda_2>\alpha_b$, we find

$$\begin{aligned}R&=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_b}\frac{\xi_0+b}{b-a}d\alpha+\int_{\alpha_b}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha\right)\\&=\frac{1}{\lambda_2-\lambda_1}\left[\int_{\lambda_1}^{\alpha_b}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}+\frac{b}{b-a}\right)d\alpha+2\int_{\alpha_b}^{\lambda_2}\frac{(1-\alpha)^2}{q\alpha(b-a)}d\alpha\right]\end{aligned}\tag{63}$$

or

$$R=\frac{b(\alpha_b-\lambda_1)}{(b-a)(\lambda_2-\lambda_1)}+\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\times\left[2\ln\left(\frac{\lambda_2^2}{\lambda_1\alpha_b}\right)+2\lambda_2(\lambda_2-4)-\alpha_b(\alpha_b-4)-\lambda_1(\lambda_1-4)\right]\tag{64}$$

For $\lambda_j\in]\alpha_b,1]$, we have

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha=\frac{2}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{(1-\alpha)^2}{q\alpha(b-a)}d\alpha\tag{65}$$

or

$$R=\frac{1}{q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\lambda_1}\right)+\lambda_2(\lambda_2-4)-\lambda_1(\lambda_1-4)\right]\tag{66}$$

For $\lambda_1\in]\alpha_b,1]$ and $\lambda_2=1$, we get

Table 4 Reliability values as depending upon bounds of the applied load

Case No.	1	2	3	4	5	6	7
λ_1	0.4	0.4	0.4	0.62	0.62	0.7	0.7
λ_2	0.5	0.65	0.7	0.64	0.7	0.9	1
R_{TH}	1	0.9796	0.9373	0.8769	0.7755	0.2829	0.1946
R_{MC}	1	0.9794	0.9374	0.8773	0.7756	0.2829	0.1945
Cov R_{MC} (%)	na	0.22	0.12	0.08	0.06	0.02	0.02

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{1}\frac{2\xi_0}{b-a}d\alpha=\frac{2}{\lambda_2-\lambda_1}\int_{\lambda_1}^{1}\frac{(1-\alpha)^2}{q\alpha(b-a)}d\alpha \quad (67)$$

or

$$R=\frac{1}{q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{1}{\lambda_1}\right)-3-\lambda_1(\lambda_1-4)\right] \quad (68)$$

Consider now $|a|<b$. We discern also seven different cases.

For $\lambda_j\in[0,\alpha_b]$,

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}d\alpha=1 \quad (69)$$

For $\lambda_1\in[0,\alpha_b]$ and $\lambda_2\in]\alpha_b,\alpha_a]$, we find

$$R=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_b}d\alpha+\int_{\alpha_b}^{\lambda_2}\frac{\xi_0-a}{b-a}d\alpha\right)=\frac{\alpha_b-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{\lambda_2-\lambda_1}\int_{\alpha_b}^{\lambda_2}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}-\frac{a}{b-a}\right)d\alpha \quad (70)$$

or

$$R=\frac{\alpha_b-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\alpha_b}\right)+\lambda_2(\lambda_2-4)-\alpha_b(\alpha_b-4)\right]-\frac{a(\lambda_2-\alpha_b)}{(b-a)(\lambda_2-\lambda_1)} \quad (71)$$

For $\lambda_1\in[0,\alpha_b]$ and $\lambda_2>\alpha_a$

$$R=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_b}d\alpha+\int_{\alpha_b}^{\alpha_a}\frac{\xi_0-a}{b-a}d\alpha+\int_{\alpha_a}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha\right)=\frac{\alpha_b-\lambda_1}{\lambda_2-\lambda_1}+\frac{1}{\lambda_2-\lambda_1}\left[\int_{\alpha_b}^{\alpha_a}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}-\frac{a}{b-a}\right)d\alpha+\int_{\alpha_a}^{\lambda_2}\frac{2(1-\alpha)^2}{q\alpha(b-a)}d\alpha\right] \quad (72)$$

or

$$R=\frac{\alpha_b-\lambda_1}{\lambda_2-\lambda_1}+\frac{a(\alpha_b-\alpha_a)}{(b-a)(\lambda_2-\lambda_1)}+\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2^2}{\alpha_a\alpha_b}\right)+2\lambda_2(\lambda_2-4)-\alpha_a(\alpha_a-4)-\alpha_b(\alpha_b-4)\right] \quad (73)$$

For $\lambda_j\in]\alpha_b,\alpha_a]$, evaluation of R reads

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{\xi_0-a}{b-a}d\alpha=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\left(\frac{(1-\alpha)^2}{q\alpha(b-a)}-\frac{a}{b-a}\right)d\alpha \quad (74)$$

or

$$R=\frac{1}{2q(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2}{\lambda_1}\right)+\lambda_2(\lambda_2-4)-\lambda_1(\lambda_1-4)\right]-\frac{a}{(b-a)} \quad (75)$$

For $\lambda_1\in[\alpha_b,\alpha_a]$ and $\lambda_2\in]\alpha_a,1]$, we get

$$R=\frac{1}{\lambda_2-\lambda_1}\left(\int_{\lambda_1}^{\alpha_a}\frac{\xi_0-a}{b-a}d\alpha+\int_{\alpha_a}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha\right) \quad (76)$$

or

$$R=\frac{1}{2(b-a)(\lambda_2-\lambda_1)}\left[2\ln\left(\frac{\lambda_2^2}{\lambda_1\alpha_a}\right)+2\lambda_2(\lambda_2-4)-\alpha_a(\alpha_a-4)-\lambda_1(\lambda_1-4)\right]-\frac{a(\alpha_a-\lambda_1)}{(b-a)(\lambda_2-\lambda_1)} \quad (77)$$

For $\lambda_j\in]\alpha_a,1]$, reliability equals

$$R=\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{2\xi_0}{b-a}d\alpha=\frac{2}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\frac{(1-\alpha)^2}{q\alpha(b-a)}d\alpha \quad (78)$$

or

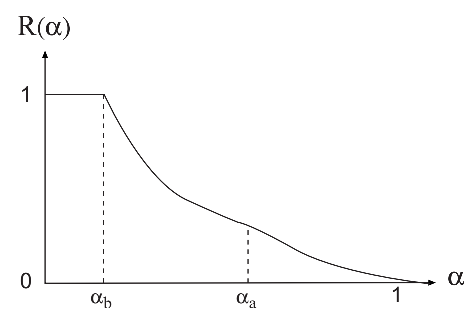

Fig. 6 Curve shape of reliability when $\alpha_b<\alpha_a$, $q=4/3$, $a=-0.1$ and $b=0.2$

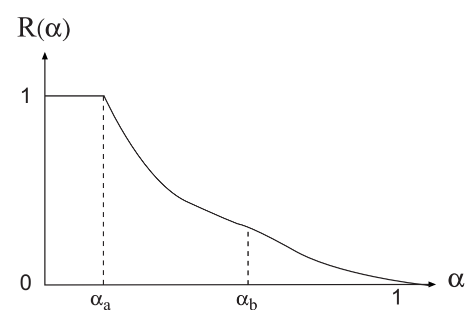

Fig. 7 Curve shape of reliability when $\alpha_a<\alpha_b$, $q=4/3$, $a=-0.2$, and $b=0.1$

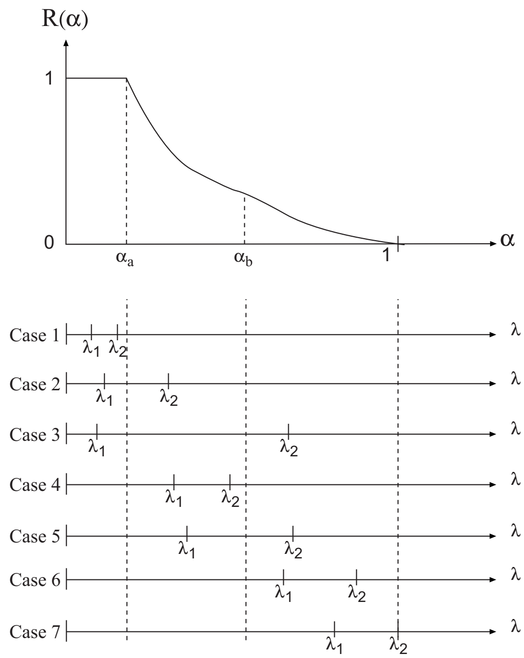

Fig. 8 Seven possible different locations of load bounds with respect to α_a, α_b, and unity

$$R = \frac{1}{q(b-a)(\lambda_2-\lambda_1)}\left[2\ \ln\left(\frac{\lambda_2}{\lambda_1}\right) + \lambda_2(\lambda_2-4) - \lambda_1(\lambda_1-4)\right] \quad (79)$$

For $\lambda_1 \in]\alpha_b, 1]$ and $\lambda_2=1$, calculations lead to

$$R = \frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{1} \frac{2\xi_0}{b-a} d\alpha = \frac{2}{\lambda_2-\lambda_1}\int_{\lambda_1}^{1} \frac{(1-\alpha)^2}{q\alpha(b-a)} d\alpha \quad (80)$$

or

$$R = \frac{1}{q(b-a)(\lambda_2-\lambda_1)}\left[2\ \ln\left(\frac{1}{\lambda_1}\right) - 3 - \lambda_1(\lambda_1-4)\right] \quad (81)$$

8 Numerical Examples and Discussion

First, we consider the case $|a|<b$, namely, $a^{(I)}=-0.1$, $b^{(I)}=0.2$, and $q=4/3$. Equations (50) and (51) provide expressions of α_a and α_b, we get $\alpha_a^{(I)}=0.6955$ and $\alpha_b^{(I)}=0.6$.

We obtain in Table 4 reliability values (Figs. 6–8) for these seven cases defined above with the theory and with the Monte Carlo method with 5×10^5 simulations with MATLAB software [12]. First, λ_j values are chosen in their respective definition areas.

The crude Monte Carlo simulation, conducted by MATLAB software [12], shows that the reliability remains unchanged whether $\xi \in [0, \xi_0]$ or $\xi \in [-\xi_0, 0]$. This is in agreement with the theoretical

Table 5 Results with PHIMECA SOFT simulations

Case No.	1	2	3	4	5	6	7
λ_1	0.4	0.4	0.4	0.62	0.62	0.7	0.7
λ_2	0.5	0.65	0.7	0.64	0.7	0.9	1
R_{form}	na	0.9345	0.9000	0.8741	0.7601	0.4599	0.4039
R_{sorm}	na	0.9590	0.9304	0.8776	0.7724	0.4565	0.3869
R_{sormHB}	na	0.9622	0.9363	0.8790	0.7813	0.4829	0.4430
R_{sormT}	na	0.9642	0.9397	0.8791	0.7826	0.4849	0.4504
R_{TH}	1	0.9796	0.9373	0.8769	0.7755	0.2829	0.1946
R^2_{smart}	na	0.9796	0.9360	0.8770	0.7740	0.2970	0.1930

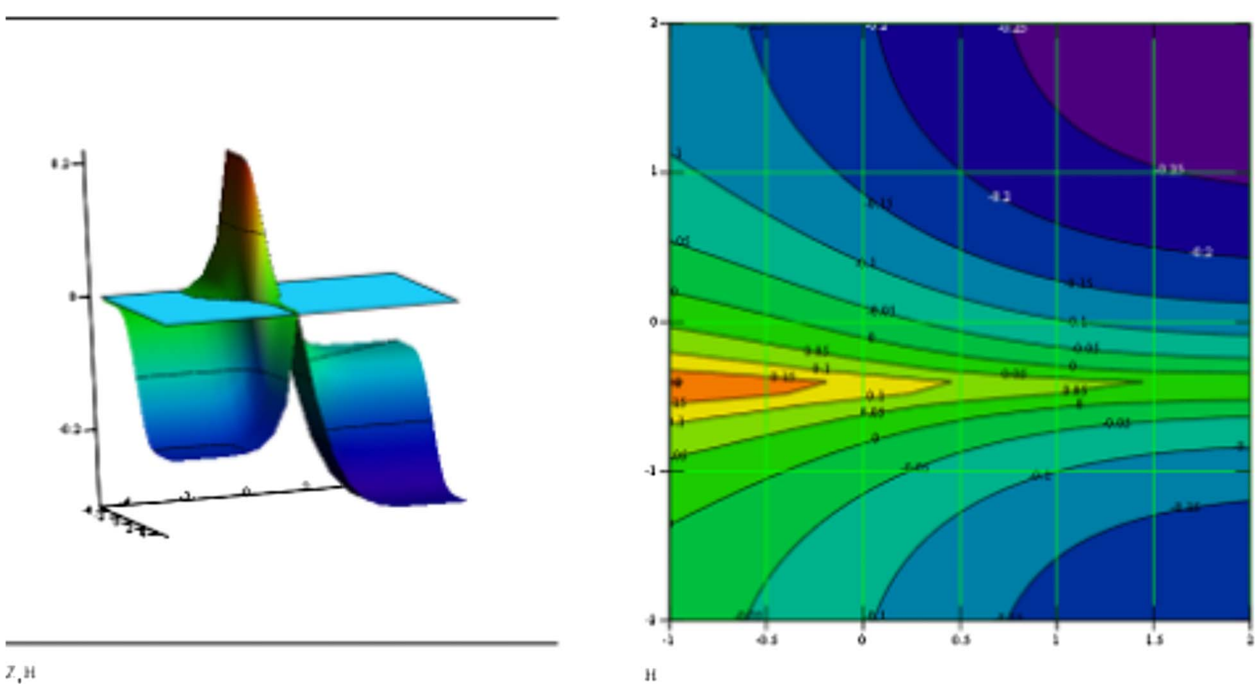

Fig. 9 Performance function for the case 6 and its isovalues

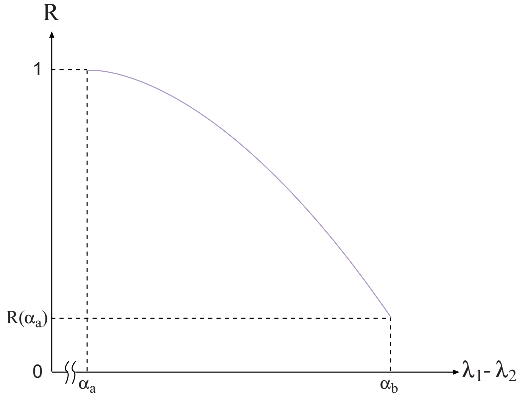

Fig. 10 Variation of reliability for the diameter $\lambda_2 - \lambda_1$ of the load variation for $a=-0.2$ and $b=0.1$

conclusion reached in Sec. 5. Calculation by PHIMECA [13] software yields the following results in Table 5. The two last cases do not show good results for the FORM/SORM calculations in comparison with the theoretical results. FORM/SORM approximations are applicable only when the state limit function owns good properties in the normed space. Results show acceptable solutions and are very economical for calculations of cases 1–5. For the two last cases, these methods are not able to follow the state limit complexity (Fig. 9). In this case, PHIMECA SOFT software builds an approximation of the state limit thanks to a generation of points based on the method of support vector machine completed by conditioned simulations depending on the successive threshold in order to limit the number of needed calculations (Table 5). It is the solution ^{2}SMART [14].

For the case $|a|>b$, we consider $a^{(II)}=-0.2$, $b^{(II)}=0.1$, and $q=4/3$ yielding $\alpha_a^{(II)}=0.6$ and $\alpha_b^{(II)}=0.6955$. Note that in the new circumstances, we have chosen

$$|a^{(II)}| = b^{(I)}, \quad |a^{(I)}| = b^{(II)} \tag{82}$$

Then by choosing the same values of λ_j, as in Table 4, we get the same reliability values. Indeed, it is remarkable that expressions of reliability for $|a|>b$ have symmetrical expressions as $|a|<b$, if conditions in Eq. (82) are satisfied.

Table 4 shows that the reliability decreases from unity to zero when either λ_j increases. Moreover, when the range $\lambda_2-\lambda_1$, i.e., the distance between λ_1 and λ_2 increases, the reliability decreases. To illustrate this property, we consider case 2 when $|a|>b$. To evaluate the reliability we fix λ_1 at zero, and let λ_2 vary from $\alpha_a=0.6$ to $\alpha_b=0.6955$. As is seen from the Fig. 10, the reliability decreases: At $\lambda_2=0.6$, the reliability equals unity, at $\lambda_2=0.65$, the reliability value is 0.9922, and finally at $\lambda_2=0.6955$, reliability equals 0.9752.

The work on the combined random initial imperfection and the random thickness variation is underway and will be reported elsewhere.

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